International Journal of Theoretical Physics, Vol. 16, No. 6 (1977), pp. 447–464

Prequantum Bundles and Projective Hilbert Geometries

CHRISTIAN GÜNTHER

Department of Mathematics, Harvard University, Cambridge, Massachusetts

Received: 3 *February* 1977

Abstract

Principal circle bundles with connection and symplectic curvature over Banach manifolds are investigated. Using results on contact manifolds alternate proofs for some results of B. Kostant are given and a symplectic structure for the total space of the corresponding principal $\mathbb{C}\setminus\{0\}$ bundle is constructed. As an example, these results are applied to the projective fibration of a complex Hilbert space. This gives close relations between the geometric formulation of classical and quantum dynamical systems. As another application, a functorial construction of the prequantization procedure of B. Kostant is given.

1. *Introduction*

In the symplectic formulation of classical Hamiltonian mechanics pure states are represented by elements of a symplectic manifold (M, ω). Observables are given by continuous functions on M. The space $\mathscr{F}M$ of all smooth observables is a Lie algebra under Poisson brackets. The symplectic maps are the isomorphism of the system, and dynamics is given by flows of Hamiltonian vector fields.

A prequantum bundle L^c over a symplectic manifold (M, ω) is a principal circle bundle over M with curvature ω. These bundles have been introduced by B. Kostant (1970) for the construction of the prequantization map, which is a representation of the Lie algebra $\mathscr{F}M$|by antisymmetric operators on a Hilbert space. B. Kostant (1970) showed that dynamics of a mechanical system (M, ω) can be expressed in terms of the prequantum bundle. This in turn may be interpreted as mechanics with phase factors. In particular, there is an isomorphism between $\mathscr{F}M$ and the Lie algebra of connection-preserving vector fields (ine infinitesimal prequantomorphisms) on the prequantum bundle L^c.

In the Hilbert-space formulation of quantum mechanics pure states are unit rays of a Hilbert space $\mathbb{H}$, i.e., elements of the complex projective Hilbert space $\hat{\mathbb{H}} = \mathbb{H}/\mathbb{C}$. Dynamics of the quantum system is given by the Schrödinger equation, which describes the flow of a Hamiltonian vector field (the Hamilton operator) on $\mathbb{H}$ with the natural symplectic structure.

In this paper we investigate prequantum bundles over possibly infinite-dimensional manifolds and show, as an important example, that the projective fibration of a Hilbert space is a prequantum bundle. This fact enables us to give a unified geometric description of classical and quantum dynamical systems.

In Sections 2 and 3 we prove some necessary results on contact manifolds and show the equivalence between torus bundles with connection and certain principal $\mathbb{C}\setminus\{0\}$ bundles. The next two sections are the main part of the paper. In Section 4 we investigate the structure of prequantum bundles over Banach manifolds. Using the results of Section 2, we give alternate proofs for some of the results of B. Kostant (1970). Moreover, we construct a symplectic form on the total space of the complex prequantum bundle such that the connection-preserving vector fields are just the invariant Hamiltonian systems.

In Section 5 we show that the projective fibration of a Hilbert space $\mathbb{H}$ defines a prequantum bundle, which induces the natural symplectic structure on $\mathbb{H}$.

We apply these results in the last two sections. In Section 6 we show the close relations between the geometric structure of classical and quantum dynamical systems, if we use the unified description in terms of prequantum bundles.

As another application we give in Section 7 a functorial construction of the prequantization procedure of B. Kostant in the category of prequantum bundles.

The differential geometric notation in this paper is, although slightly modified, based on the book of Abraham and Marsden (1967). In particular we use the following symbols: TM is the tangent bundle of M; Tf is the tangent map of $f{:}M \to N$ smooth; $T'M$ is the cotangent bundle (we use the $\|\cdot\|$ topology on the dual space of a Banach space); $\mathscr{F}M$ is the ring of real smooth functions on M; $\mathscr{X}M$ is the Lie algebra of smooth vector fields on M; $\mathscr{X}'M = \Omega^1 M$ are the one-forms on M; $\mathbb{L}_X$ is the Lie derivative; $\lrcorner$ is the inner product; $\mathscr{H}M$ are the Hamiltonian vector fields on M; LG is the Lie algebra of the Lie group G; hor = horizontal, ver = vertical, inv = invariant, equ = equivariant, $\dot{\mathbb{C}} = \mathbb{C}\setminus\{0\}$, and $\dot{\mathbb{R}} = \mathbb{R}\setminus\{0\}$.

2. *Contact Manifolds*

Contact structures are needed for the construction of the prequantization map. In this section some fundamental facts about contact manifold are presented.

2.1. *Definition.* Let M be a Banach manifold, $\vartheta \in \Omega^1 M$ a one-form on M. Define the presymplectic form $\omega := -d\vartheta$ and the induced musicalic morphisms $\omega^b: TM \to T'M, v_m \mapsto \omega(v_m, \cdot)$. ϑ is called a *contact form* iff for all $m \in M$.

(a) $\dim \operatorname{Ker} \omega^b(m) = \operatorname{codim} \operatorname{Im} \omega^b(m) = 1$
(b) $\operatorname{Ker} \vartheta \oplus_M \operatorname{Ker} \omega = TM$

For any contact manifold (M, ϑ) the *characteristic bundle* $CM := \operatorname{Ker} \omega$ defines a one-dimensional fibration of M, the *characteristic fibration.* Sections in CM are called characteristic vector fields. Denote by $\mathscr{C}M$ the local one-dimensional $\mathscr{F}M$ module of characteristic vector fields.

2.2. *Remark.* There is a vector field $Z \in \mathscr{C}M$ uniquely defined by

$$Z \lrcorner\, \omega = 0 \quad \text{and} \quad Z \lrcorner\, \vartheta = 1$$

Z is called the *Cartan vector field* and is a base of $\mathscr{C}M$.

For two contact manifolds (M_i, ϑ_i), $i = 1, 2$ *contact morphisms* are smooth maps $f: M_1 \to M_2$ with $f^*\vartheta_2 = \vartheta_1$. Note that contact morphisms preserve characteristic fibers and $Tf(CM_1) \subset CM_2$ for any contact map f. In particular $Tf(Z(m_1)) = Z(f(m_1))$ for $m_1 \in M_1$. Denote by *CtMf* the category of contact manifolds with contact morphisms.

2.3. *Notation.* Let (M, ϑ) be a contact manifold. $\operatorname{Cont}(M, \vartheta) := \{f \in \operatorname{Diff}(M) \mid f^*\vartheta = \vartheta\}$ the (special) *contact group.* $\mathscr{P}M := \{X \in \mathscr{X}M \mid \mathbb{L}_X \vartheta = 0\}$ the (special) *contact vector fields.* $\mathscr{F}hM := \{f \in \mathscr{F}M \mid \mathbb{L}_X f = 0$ for all $X \in \mathscr{C}M\}$.

Remark: (a) $\mathscr{P}M$ is a Lie subalgebra of $\mathscr{X}M$, and we can interpret $\mathscr{P}M$ as the formal "Lie algebra" of $\operatorname{Cont}(M, \vartheta)$, since any $X \in \mathscr{P}M$ has a flow of contact isomorphisms.

(b) $\mathscr{F}hM$ consists of all real functions on M that are constant on the characteristic fibers of M. For $f \in \mathscr{F}M$ we have $f \in \mathscr{F}hM$ iff for all $m \in M$: $df_m \in \operatorname{Im} \omega^b(m)$.

2.4. *Definition.* Let (M, ϑ) be a contact manifold and $f, g \in \mathscr{F}hM$.

(a) $G_f \in \mathscr{X}M$ is uniquely determined by $G_f \lrcorner\, \omega = df$ and $G_f \lrcorner\, \vartheta = 0$.

(b) $P_f \in \mathscr{X}M$ is uniquely determined by $P_f \lrcorner\, \omega = df$ and $P_f \lrcorner\, \vartheta = f$.

(c) The *Poisson brackets* of f and g are defined by

$$\{f, g\} := G_f \lrcorner\, G_g \lrcorner\, \omega$$

2.5. *Remarks.* (a) For $f \in \mathscr{F}hM$ $\mathbb{L}_{G_f}\omega = dG_f \lrcorner\, \omega = 0$ so G_f has a presymplectic flow. G_f is the "Hamiltonian" vector field of f.

(b) We have the formula $P_f = G_f + f \cdot Z$

(c) $d\{f, g\} = [G_f, G_g] \lrcorner\, \omega$.

2.6. *Theorem.* Let (M, ϑ) be a contact manifold. The map $P.$ defined by 2.4(b) is an isomorphism of Lie algebras $P.: \mathscr{F}hM \to \mathscr{P}M$ with $\cdot \lrcorner\, \vartheta$ $(X \mapsto X \lrcorner\, \vartheta)$ as inverse map.

Proof. for $f \in \mathscr{F}hM$ we have $\mathbb{L}_{P_f}\vartheta = P_f \lrcorner\, d\vartheta + dP_f \lrcorner\, \vartheta = 0$, so $P_f \in \mathscr{P}M$. $P_f \lrcorner\, \vartheta = f$ by definition. To show $P_{X \lrcorner \vartheta} = X$ for all $X \in \mathscr{P}M$ we use the decomposition of 2.1.(b): $P_{X \lrcorner \vartheta} \lrcorner\, \vartheta = X \lrcorner\, \vartheta$ by definition and $P_{X \lrcorner \vartheta} \lrcorner\, d\vartheta = -d(X \lrcorner\, \vartheta) = -\mathbb{L}_X \vartheta + X \lrcorner\, d\vartheta = X \lrcorner\, d\vartheta$, since $X \in \mathscr{P}M$. Thus X and $P_{X \lrcorner \vartheta}$

have the same components. This proves $\cdot \lrcorner\, \vartheta$ to be the inverse of $P.$. By definition $P_{\{f,g\}} \lrcorner\, \vartheta = \{f, g\}$ so $\cdot \lrcorner\, \vartheta$ is a Lie algebra morphism.

$$[P_f, P_g] \lrcorner\, \vartheta = \mathbb{L}_{P_f} P_g \lrcorner\, \vartheta = \{f, g\} = P_{\{f,g\}} \lrcorner\, \vartheta$$

and

$$[P_f, P_g] \lrcorner\, \omega = dP_f \lrcorner\, P_g \lrcorner\, \omega = d\{f, g\} = P_{\{f,g\}} \lrcorner\, \omega$$

So $P_{\{f,g\}} = [P_f, P_g]$. This proves 2.6.

For a contact manifold (M, ϑ) we define $M_{\mathbb{R}^+} := \mathbb{R}^+ \times M$ and $p: M_{\mathbb{R}^+} \to M$, $s: M_{\mathbb{R}^+} \to \mathbb{R}^+$ the natural projections. Then $(M_{\mathbb{R}^+}, p, M, \mathbb{R}^+)$ is a principal $(\mathbb{R}^+, \cdot)$ bundle with $d(\log s)$ as the natural connection form. The two-form $-d(s \cdot p^*\vartheta) = -ds \wedge p^*\vartheta - s \cdot p^*d\vartheta$ is nondegenerate and $\mathbb{R}^+$ equivariant. So we have the following:

2.7. *Proposition.* $(\mathbb{R}^+ \times M, -d(s \cdot p^*\vartheta))$ is a symplectic manifold. If $f: (M_1, \vartheta_1) \to (M_2, \vartheta_2)$ is a contact morphism, $1_{\mathbb{R}^+} \times f: (\mathbb{R}^+ \times M_1, -d(s_1 \cdot p_1^*\vartheta_1)) \to (\mathbb{R}^+ \times M_2, -d(s_2 \cdot p_2^*\vartheta_2))$ is a $\mathbb{R}^+$-equivariant symplectic (bundle) map. Thus the correspondence $M \mapsto M_{\mathbb{R}^+}, f \mapsto 1_{\mathbb{R}^+} \times f$ is a covariant functor from the category of contact manifolds into the category of symplectic manifolds.

Moreover, for $X \in \mathscr{X}M$ there is a unique continuation of X to a horizontal $\mathbb{R}^+$-invariant vector field $\tilde{X} \in \mathscr{X}(\mathbb{R}^+ \times M)$. This defines a Lie algebra isomorphism $\tilde{\ }: \mathscr{X}M \to \mathscr{X}^{\text{inv}}_{\text{hor}}(\mathbb{R}^+ \times M)$ and $\tilde{\ }$ is the Lie algebra morphism corresponding to the group isomorphism $\text{Diff}(M) \to \text{Aut}^{\text{equ}}(\mathbb{R}^+ \times M), f \mapsto 1_{\mathbb{R}^+} \times f$.

2.8. *Proposition.* Let $X \in \mathscr{X}M$. Then $\tilde{X}$ is a Hamiltonian vector field on $(\mathbb{R}^+ \times M, -d(s \cdot p^*\vartheta))$ iff $X \in \mathscr{P}M$. Thus $\tilde{\ }$ defines a Lie algebra isomorphism $\mathscr{P}M \to \mathscr{H}^{\text{inv}}\mathbb{R} \times M$ from contact vector fields into the invariant Hamiltonian vector fields.

For $f, g \in \mathscr{F}hM$ we have $\{s \cdot f \circ p, s \cdot g \circ p\} = s \cdot \{f, g\} \circ p$, where the brackets on the left are the usual Poisson brackets on the symplectic manifold $\mathbb{R}^+ \times M$ and the brackets on the right are given by 2.4.

2.9. *Proposition.* Let $\mathscr{F}h^{\text{equ}}\mathbb{R}^+ \times M$ be all smooth functions $f \in \mathscr{F}\mathbb{R}^+ \times M$ that are $\mathbb{R}^+$ equivariant and have the property $\tilde{X} \lrcorner\, df = 0$ for all $X \in \mathscr{C}M$. Then the map $\mathscr{F}hM \to \mathscr{F}h^{\text{equ}}\mathbb{R}^+ \times M, f \mapsto s \cdot f \circ p$ is a Lie algebra isomorphism and we have $\tilde{P}_f \lrcorner -d(s \cdot p^*\vartheta) = d(s \cdot f \circ p)$, i.e., $s \cdot f \circ p$ is the Hamiltonian of $\tilde{P}_f$.

Remark. Having $Z = P_1$ we get $\tilde{Z} \lrcorner -d(s \cdot p^*\vartheta) = ds$. For $f \in \mathscr{F}hM$ $G_f \lrcorner -d(s \cdot p^*\vartheta) = d(f \circ p) \cdot s$.

3. *Line Bundles and Circle Bundles*

In the following, some relations between circle bundles and Hermitian line bundles are investigated. Standard facts concerning connections on principal and vector bundles are found in Greub et al. (1973).

3.1. *Definition.* A *line bundle* $L = (L, \pi, M)$ over a Banach manifold M is a one-dimensional complex vector bundle over M. Any line bundle (L, π, M) is associated with a principal $\dot{\mathbb{C}}$ bundle $\dot{L} = (\dot{L}, \dot{\pi}, M, \dot{\mathbb{C}})$, where $\dot{\mathbb{C}} := \mathbb{C}\setminus\{0\}$, the multiplicative group of nonzero complex numbers. Because $L = \dot{L} \times_{\dot{\mathbb{C}}} \mathbb{C}$ and $\dot{L} = \dot{L} \times_{\dot{\mathbb{C}}} \dot{\mathbb{C}}$, we get: $\dot{L} = L\backslash$zero section.

> 3.2. *Proposition.* There is a one-to-one correspondence between line bundles L and principal $\dot{\mathbb{C}}$ bundles $\dot{L}$ over a Banach manifold M.

If $(L, \langle\cdot,\cdot\rangle)$ is an Hermitian line bundle, $(L, \langle\cdot,\cdot\rangle)$ is, as a Σ bundle, associated with a principal circle bundle $L^c = (L^c, \pi^c, M, \mathbb{T})$ over M, where $\mathbb{T}$ is the one-dimensional torus. For any positive $r \in \mathbb{R}$ denote by $S^r(L, \langle\cdot,\cdot\rangle)$ the r-sphere bundle of $(L, \langle\cdot,\cdot\rangle)$: $S^r(L, \langle\cdot,\cdot\rangle) = \{x_m \in L \mid \langle x_m, x_m\rangle = r^2\}$. Since $L = L^c \times_{\mathbb{T}} \mathbb{C}$ and $S^r(L, \langle\cdot,\cdot\rangle) = L^c \times_{\mathbb{T}} S^r(\mathbb{C})$, we have for any $r \in \mathbb{R}^+$ an isomorphism of fibre bundles $S^r(L) \to L^c$ induced by the isomorphism between $S^r(\mathbb{C})$ and $\mathbb{T}$

Note: If we identify $\mathbb{T}$ with $S^r(\mathbb{C})$, the unity of $\mathbb{T}$ is represented by the complex number r.

> 3.3. *Proposition.* There is a one-to-one correspondence between Hermitian line bundles $(L, \langle\cdot,\cdot\rangle)$ and principal circle bundles over a Banach manifold M.

For any $r \in \mathbb{R}$ there are isomorphisms $L^c \to S^r(L)$ inducing inclusions

$$L^c \overset{i}{\hookrightarrow} L \overset{i}{\hookrightarrow} L$$

(If $q : L^c \times \mathbb{C} \to L$ is the natural projection, we have $i_r = q|_{L^c \times S^r(\mathbb{C})}$.)

3.4. *Notation.* Let $P = (P, \pi, M, G)$ be a principal G bundle over the Banach manifold M with Abelian group G. The action of $GA : G \times P \to P$ induces a Lie algebra morphism $\hat{}: LG \to \mathscr{X}P, \eta \mapsto \hat{\eta}$ with $\hat{\eta}(x_m) = (d/dt)\,[x_m \cdot \exp(t\cdot\eta)]_{t=0}$ for $\eta \in LG$ and $x_m \in P$. $\hat{\eta}$ is called the *fundamental vector field* induced by $\eta \cdot \hat{\eta}$ is vertical and G invariant.

A LG valued one-form $\alpha \in \Omega^1(P, LG)$ is a *connection form* on P iff $\mathbb{L}_{\hat{\eta}}\alpha = 0$ and $\hat{\eta} \lrcorner\, \alpha = \eta$ for all $\eta \in LG$. Hor$P$:= Kerα is the horizontal bundle of P and we have Hor$P \oplus_M$ Ver$P = TP$. We call $H: TP \to$ Hor$P$ and $V = H - 1: TP \to$ Ver$P$ the horizontal and vertical projections. On any associated vector bundle $E = (E, \rho, M)$ of P, α induces a covariant derivative $\nabla_\cdot : \Gamma^\infty E \to \mathrm{Hom}(TM, E)$ and a corresponding horizontal bundle of E.

A Hermitian structure $\langle\cdot,\cdot\rangle$ on a complex vector bundle E with complex connection $\nabla_\cdot$ is called $\nabla_\cdot$ *affine* (or $\nabla_\cdot$ *invariant*) iff for all smooth sections $\sigma, \sigma' \in \Gamma^\infty E$: $d\langle\sigma, \sigma'\rangle = \langle\nabla_\cdot\sigma, \sigma'\rangle + \langle\sigma, \nabla_\cdot\sigma'\rangle$. The real part Re$\langle\cdot,\cdot\rangle$ of a Hermitian structure is a Riemannian, and the imaginary part Im$\langle\cdot,\cdot\rangle$ is a symplectic structure on E. These structures are connected by the complex structure Re$\langle x, y\rangle$ = Im$\langle ix, y\rangle$ for all $x, y \in E$. From the $\mathbb{C}$ linearity of $\nabla_\cdot$ $\langle\cdot,\cdot\rangle$ is $\nabla_\cdot$ affine iff Re$\langle\cdot,\cdot\rangle$ is $\nabla_\cdot$ affine, i.e., iff $\nabla_\cdot$ is a Riemannian connection.

By standard results of Riemannian geometry (Flaschel and Klingenberg, 1972; Greub et al., 1973) we have that $\langle\cdot,\cdot\rangle$ is $\nabla_\cdot$ affine, iff the corresponding horizontal subspaces $\mathrm{Hor}_x E$, $x \in E$ of $T_x E$ are tangent to the sphere bundle $S^r(E)$, $r = |x|$ with $|x| := \langle x, x\rangle^{1/2}$. Thus we obtain the following:

3.5. *Proposition.* Let (L^c, α^c) be a principal circle bundle with connection and $(L, \nabla_.)$ the corresponding line bundle with covariant derivative. For a Hermitian structure $\langle \cdot, \cdot \rangle$ on L the following statements are equivalent:
1. $\langle \cdot, \cdot \rangle$ is $\nabla_.$ affine.
2. $\mathrm{Hor}_x L \subset TS^{|x|}(L)$ for all $x \in L$
3. For all $r \in \mathbb{R}^+$ the inclusion $i_r: L^c \hookrightarrow L$ is connection preserving:

$$\mathrm{T}i_r(\mathrm{Hor}_x L^c) \subset \mathrm{Hor}_{i_r(x)} L \qquad \text{for all } x \in L^c$$

In that case $\mathrm{Hor}L|_{S^r(L)}$ is a horizontal bundle of $S^r(L)$ and $\mathrm{T}i_r(\mathrm{Hor}L^c) = \mathrm{Hor}L|_{S^r(L)}$.

For the proof of (3) note that $i_r = q|_{L^c \times S^r(\mathbb{C})}$, where $q: L^c \times \mathbb{C} \to L$ is the natural projection.

3.6. *Corollary.* There is a one-to-one correspondence between principal circle bundles with connection (L^c, α^c) and line bundles with covariant derivative and affine Hermitian structure $(L, \nabla_., \langle \cdot, \cdot \rangle)$ over a Banach manifold M.

Moreover, for any positive $r \in \mathbb{R}^+$ there are imbeddings of bundles $i_r: L^c \hookrightarrow \dot{L} \hookrightarrow L$ and $\mathrm{T}i_r$: $\mathrm{Hor}L^c \hookrightarrow \mathrm{Hor}\dot{L} \hookrightarrow \mathrm{Hor}L$. $r \in \mathbb{R}^+$ gives rise to an isomorphism

$$\dot{L} \to \mathbb{R}^+ \times L^c, x_m \mapsto \left(|x_m|, \frac{x_m}{|x_m|}\, r\right)$$

Thus by 3.6 $\mathrm{Hor}\dot{L} = \cup_{\mathbb{R}^+} \mathrm{Hor}S^r(L)$ and we can identify $\dot{L} = \mathbb{R}^+ \times L^c$. Note that $(\dot{L}, p, L^c, \mathbb{R}^+)$ is a (trivial) principal $(\mathbb{R}^+, \cdot)$ bundle with $d(\log s)$ as natural connection form, where $s: \dot{L} \to \mathbb{R}^+, x \mapsto |x|$. The covering $\mathbb{R} \to \mathbb{T}$ and the imbedding $\mathbb{T} \hookrightarrow \dot{\mathbb{C}}$ induce a Lie-algebra homomorphism $\mathbb{R} = L\mathbb{T} \to L\dot{\mathbb{C}} = \mathbb{C}, t \mapsto \lambda \cdot t$, where $2\pi/\lambda$ is the period of the covering.

3.7. *Theorem.* Let (L^c, α^c) be a principal circle bundle with connection form α^c and let $(L, \nabla_., \langle \cdot, \cdot \rangle)$ be the corresponding line bundle with covariant derivative $\nabla_.$ and $\nabla_.$ affine Hermitian structure $\langle \cdot, \cdot \rangle$. Then the unique connection form α on $\dot{L}$, which corresponds (by association) to $\nabla_.$, is given by

$$\alpha_x = d(\log |x|) + (i/\lambda) \cdot p^*\alpha^c_x \qquad \text{for } x \in \dot{L}$$

Conversely, if $\dot{L}$ is a principal $\dot{\mathbb{C}}$ bundle with a connection form α and if L is the corresponding line bundle with covariant derivative $\nabla_.$, then there exists a $\nabla_.$ affine Hermitian structure $\langle \cdot, \cdot \rangle$ on L if and only if $\mathrm{Re}\alpha$ is exact. In this case $\mathrm{Im}\alpha$ defines (by restriction) a connection form α^c on the corresponding principal circle bundle $L^c \cong S^1(L)$.

Proof. We have only to show, that α is a connection form on $\dot{L}$ and that $\mathrm{Ker}\alpha = \cup_{r \in \mathbb{R}^+} Ti_r(\mathrm{Hor} L^c)$. Let $c = a + ib \in \mathbb{C} = L\dot{\mathbb{C}}$. Then $\hat{c}_x = (d/dt)(e^{ct} \cdot x)_{t=0}$ for all $x \in \dot{L}$ and hence

$$\hat{c} \lrcorner \alpha_x = \frac{d}{dt} \log \langle e^{ct}x, e^{ct}x \rangle^{1/2}|_{t=0} + \frac{i}{\lambda} \cdot p^*\alpha^c \left(\frac{d}{dt} e^{ct} \cdot x \right) |_{t=0}$$

and by logarithmic differentiation

$$= \frac{d}{dt} (e^{at} \langle x, x \rangle^{1/2}) / e^{at} \langle x, x \rangle^{1/2} |_{t=0} + \frac{i}{\lambda} \alpha^c \{ \lambda \cdot \hat{b} [p(x)] \}$$

$$= a + ib$$

since α^c is a connection form of L^c. The rest of the proof follows from the decomposition $T\dot{L} \cong T\mathbb{R}^+ \times TL^c \cong T\mathbb{R}^+ \times \mathrm{Ver} L^c \oplus_M \mathrm{Hor} L^c$ and from 3.6.

Remark. In the following we take $\lambda = 1$. In prequantization theory we have usually $\lambda = \hbar = h/2\pi$.

3.8. *Proposition.* Let (L_i^c, α_i^c), $i = 1, 2$ be principal circle bundles over M_i and $(L_i, \nabla^i, \langle \cdot, \cdot \rangle_i)$ the corresponding line bundles as above. Then by continuation (or, respectively, restriction) there is a one-to-one correspondence between connection-preserving $\mathbb{T}$-equivalent bundle maps $f^c: L_1^c \to L_2^c$, $(f^*\alpha_2^c = \alpha_1^c)$ and connection-preserving $\dot{\mathbb{C}}$-equivariant isometric maps $f: L_1 \to L_2$, $f^*\nabla^2 = f^*\nabla^1$ and $f^*|\cdot|_2 = |\cdot|_1$.

3.9. *Proposition.* Let (L^c, α^c) and $(L, \nabla, \langle \cdot, \cdot \rangle)$ be as above. Then there is a one-to-one correspondence between $\mathbb{T}$-invariant vector fields $X^c \in \mathscr{X}^{\mathrm{inv}} L^c$, that preserve the connection $\mathbb{L}_{X^c} \alpha^c = 0$ and $\dot{\mathbb{C}}$-invariant vector fields $X \in \mathscr{X}^{\mathrm{inv}} \dot{L}$, that preserve connection $\mathbb{L}_X c = 0$ and that preserve the metric $\mathbb{L}_X |\cdot| = 0$. This correspondence is an isomorphism of Lie algebras.

The proofs of 3.8 and 3.9 are, by 3.6, trivial.

3.10. *Remarks.* If $Z = \hat{1}^T$ is the fundamental vector field of L^c with 1^T the unity of $\mathbb{T}$, then the unique continuation by $L^c \hookrightarrow \dot{L}$ of Z has the form $Z = i\hat{h}$.

4. *Prequantum Bundles*

4.1. *Definition:* Let (M, ω) be a symplectic Banach manifold. A *prequantum bundle* (PQB) over (M, ω) is a principal circle bundle (L^c, α^c) with connection form over M such that

$$\omega = -\mathrm{curv}\, \alpha^c \qquad (\text{i.e., } d\alpha^c = -\pi^{c*}\omega)$$

The corresponding line bundle with connection and affine Hermitian structure will be called a (complex) PQB.

4.2. *Remarks.* Complex prequantum bundles were introduced and investigated by B. Kostant (1970) for the construction of the prequantization map.

The results 4.4, 4.7, and 4.8 of this section are due to B. Kostant, but we give other (partially simpler) proofs using results on contact manifolds. Because classification of bundles by Chern's classes does not require finite dimensionality of the base manifold (Vaisman, 1973), the results of B. Kostant concerning the existence of PQB's over a symplectic manifold remain valid in the infinite-dimensional case. So we have the following theorem:

> 4.2. *Theorem* (B. Kostant, 1970). A prequantum bundle over a symplectic manifold (M, ω) exists iff ω is integral. If ω is integral, the class of all PQB's over (M, ω) can be parametrized by the characters of the fundamental group of M. In particular, if M is simply connected, there is one and only one PQB over (M, ω) if ω is integral.

4.3. *Definition.* Morphisms and infinitesimal morphisms of PQB's are defined as in 3.8 and 3.9, respectively. In particular,

$$\operatorname{Preq}(L^c, \alpha^c) := \{f \in \operatorname{Aut} L^c \mid f^*\alpha^c = \alpha^c, f\ \mathbb{T}\text{equivariant}\}$$

$$\mathscr{P}L^c := \{X \in \mathscr{X}L^c \mid \mathbb{L}_X \alpha^c = 0, X\,\mathbb{T} \text{ invariant}\}$$

$$\operatorname{Spl}^{L^c}(M, \omega) := \{g \in \operatorname{Diff}(M) \mid g \text{ is base map of a suitable } f \operatorname{Preq}(L^c, \alpha^c)\}$$

$$\begin{array}{ccc} L^c & \xrightarrow{\quad f \quad} & L^c \\ \downarrow & & \downarrow \\ M & \xrightarrow{\quad g \quad} & M \end{array}$$

Remark. By 3.8 and 3.9 $\operatorname{Preq}(L^c, \alpha^c)$ is isomorphic to the group of connection-preserving isometric bundle automorphisms of the corresponding complex PQB $(L, \nabla_., \langle \cdot, \cdot \rangle)$, and $\mathscr{P}L^c$ is isomorphic to the Lie algebra of connection-preserving $\mathbb{C}$-invariant vector fields with isometric flow.

Remark. Since elements of $\operatorname{Preq}(L^c, \alpha^c)$ preserve curvature, we have $\operatorname{Spl}^{L^c}(M, \omega) \subset \operatorname{Spl}(M, \omega)$.

If M is simply connected $\operatorname{Spl}^{L^c}(M, \omega) = \operatorname{Spl}(M, \omega)$ (Kostant, 1970).

> 4.4. *Theorem* (Kostant, 1970). For a PQB (L^c, α^c) over (M, ω) the following sequence of groups is exact and defines a central extension of $\operatorname{Spl}^{L^c}(M, \omega)$ by $\mathbb{T}$:
>
> $$0 \to \mathbb{T} \overset{i}{\hookrightarrow} \operatorname{Preq}(L^c, \alpha^c) \overset{\hat{}}{\to} \operatorname{Spl}^L(M, \omega) \to 0$$
> $$c \mapsto i_c \qquad\qquad f \mapsto \hat{f}$$
>
> where $i_c(x) := c \cdot x$ is the natural action of $\mathbb{T}$ on L^c and $\hat{f}$ is the base map of f.

Proof. This is an easy property of any principal G bundle with Abelian G.

> 4.5. *Proposition:* Let (L^c, α^c) be a PQB over (M, ω). Then (L^c, α^c) is a contact manifold.

Proof. $\operatorname{Ker} \alpha^c = \operatorname{Hor} L^c$ and $\operatorname{Ker}(-d\alpha^c) = \operatorname{Ver} L^c$ since $d\alpha^c$ is horizontal and $d\alpha^c = -\pi^{c*}\omega$ is nondegenerate on horizontal subspaces.

4.6. *Corollary.* Using the definitions of Section 2 we obtain the following interpretation of the objects introduced in Section 2:
contact manifold = total space L^c
contact form = connection form α^c
presymplectic form = $-d\alpha^c = \pi^{c*}\omega$
characteristic bundle = vertical bundle $\operatorname{Ver} L^c$
characteristic fibers = fibers of the bundle L^c
quotient L^c/characteristic fibration = base manifold M

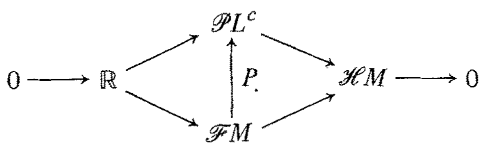
Remark. Let $f \in \operatorname{Diff}(L^c)$ be connection preserving: $f^*\alpha^c = \alpha^c$. Then $f^* \operatorname{Ker} d\alpha^c = \operatorname{Ker} d\alpha^c$, so f is fiber preserving. Moreover, $f^*Z = Z$ since Z is uniquely determined by α^c, so f is also $\mathbb{T}$ equivariant. Therefore $\operatorname{Preq}(L^c, \alpha^c) = \{f \in \operatorname{Diff}(L^c) \mid f^*\alpha^c = \alpha^c\}$. So we get additional correspondences:

$\operatorname{Cont}(L^c, \alpha^c) = \operatorname{Preq}(L^c, \alpha^c)$
$\mathscr{P}L^c$ (as defined in 2.3) = $\mathscr{P}L^c$ (as defined in 4.3)
$Z = P_1$ $\quad$ $Z = \hat{1}$ $\quad$ (1 = unity of $\mathbb{T}$)
$\mathscr{F}hM$ $\quad$ $\mathscr{F}M$

4.7. *Theorem* (Kostant, 1970). Let (L^c, α^c) be a PQB over (M, ω). Then $P\colon \mathscr{F}M \to \mathscr{P}L^c, f \mapsto P_f = G_f + f \circ \pi^c \cdot Z$ is an isomorphism of Lie algebras.

Proof. This is a trivial consequence of 2.6.

4.8. *Corollary* (B. Kostant, 1970). The following diagram of Lie algebras is commutative and has exact lines:



Application of 2.7 and 3.6 gives the following:

4.9. *Proposition.* Let (L^c, α^c) be a PQB over (M, ω) and $(\dot{L}, \alpha)$ the corresponding complex PQB. Denote by $p: \dot{L} \to L^c$ and $s = |\cdot|: \dot{L} \to \mathbb{R}^+$ the natural projections given by $\dot{L} \cong \mathbb{R}^+ \times L^c$. Then

$$\Omega := -d(s^2 \cdot p^*\alpha^c)$$

is a symplectic form on $\dot{L}$.

4.10. *Corollary.* Let $\operatorname{Spl}(\dot{L}, \Omega)$ be the symplectic group of $(\dot{L}, \Omega)$. Then $\operatorname{Preq}(\dot{L}, \alpha) \subset \operatorname{Spl}(L, \Omega)$ and for any $F \in \operatorname{Spl}(L, \Omega)$ we have $F \in \operatorname{Preq}(L, \alpha)$ iff F is fiber preserving and $\dot{\mathbb{C}}$ equivariant.

4.11. *Corollary.* Denote by $\mathscr{H}\dot{L}$ the Lie algebra of global Hamiltonian vector fields on (L, Ω), and by $\mathscr{H}^{\mathrm{inv}}\dot{L}$ all $\dot{\mathbb{C}}$ invariant Hamiltonian vector fields. Then

$$\mathscr{P}\dot{L} = \mathscr{H}^{\mathrm{inv}}\dot{L}$$

4.12. *Corollary.* Define for $f \in \mathscr{F}M$ $\tilde{f} \in \mathscr{F}\dot{L}$ by

$$\tilde{f}(x) = \langle x, x\rangle \cdot f \circ \dot{\pi}(x) = s^2(x) \cdot f \circ \dot{\pi}(x) \qquad \text{for all } x \in \dot{L}$$

Then $\tilde{f}$ is the Hamiltonian function of $P_f \in \mathscr{P}\dot{L} \subset \mathscr{H}\dot{L}$. Moreover the flows of $P_f \in \mathscr{H}\dot{L}$ and $df^{\#} \in \mathscr{H}M$ are $\dot{\pi}$-related and $\tilde{\ }: \mathscr{F}M \to \mathscr{F}\dot{L}$ is a Lie algebra morphism. ($\mathscr{F}\dot{L}$ is by Poisson brackets defined by Ω a Lie algebra).

The proofs of 4.10–4.12 follow directly from 2.7–2.9.

4.13. *Remark.* (a) $\Omega = -d(s^2 \cdot p^*\alpha^c)$ is not a $\dot{\mathbb{C}}$-invariant form. The form $-d(s \cdot p^*\alpha^c)$ defines a $\dot{\mathbb{C}}$-invariant symplectic structure on $\dot{L}$. The results of 4.10, 4.11 and 4.12 are also valid for this invariant symplectic form (if we define $\tilde{f} = s \cdot f \circ \dot{\pi}$).

(b) If we identify $\mathrm{Ver}\,\dot{L}$ with $\dot{\pi}^*L$, we have $\Omega|_{\mathrm{Ver}\dot{L} \times \mathrm{Ver}\dot{L}}$ corresponds to the imaginary part of the Hermitian structure $\langle\cdot, \cdot\rangle$ on L, or, using $\mathrm{Ver}\,\dot{L} = \dot{L} \times \mathbb{C}$, to the natural symplectic structure on $\dot{\mathbb{C}}$. This is true only for Ω and not for the $\dot{\mathbb{C}}$-invariant symplectic form of 4.13 (a). Therefore Ω seems to be more natural than the invariant symplectic form.

By the isomorphy of $\mathrm{Ver}\,\dot{L}$ and $\dot{\pi}^*L$ the complex structure on L and the Hermitian metric induce a complex structure j^v and a Hermitian metric $\langle\cdot, \cdot\rangle^v$ on $\mathrm{Ver}\,\dot{L}$ and the imaginary part of $\langle\cdot, \cdot\rangle^v$ is $\Omega|_{\mathrm{Ver}\dot{L} \times \mathrm{Ver}\dot{L}}$.

Assume there exists an almost complex structure j on the symplectic manifold M, such that (M, j, H) becomes an almost Kaehlerian manifold, where H is defined by $H(x, y) = \dot{L} \cdot \omega(x, y) + \omega(jx, y)$ for all $x, y \in TM$. Then by horizontal lifting $\mathrm{Hor}\,\dot{L}$ becomes an almost Kaehlerian vector bundle. Define

$$J := j^v + \dot{\pi}^*j \qquad \text{and} \qquad \langle\!\langle\cdot, \cdot\rangle\!\rangle := \langle\cdot, \cdot\rangle^v + \dot{\pi}^*H.$$

Then $(\dot{L}, J, \langle\!\langle\cdot, \cdot\rangle\!\rangle)$ is an almost Kaehlerian manifold. So we have the following:

4.14. *Proposition.* Let $(M, j, \langle\cdot, \cdot\rangle_M)$ be an almost Kaehlerian manifold and $\omega := \mathrm{Im}\langle\cdot, \cdot\rangle_M$ be integral. Assume (L^c, α^c) to be a PQB over (M, ω). Then the complex PQB $\dot{L}$ corresponding to L^c becomes in a natural way an almost complex manifold.

5. *Projective Hilbert Spaces*

In this section we give an important example for a prequantum bundle over an infinite-dimensional manifold: the Hopf fibration of a complex Hilbert space.

Notation. Let $(\mathbb{H}, \langle\!\langle\cdot, \cdot\rangle\!\rangle)$ be a complex Hilbert space. $\dot{\mathbb{H}} := \mathbb{H}\setminus\{0\}$. $\Omega :=$

$\text{Im}\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is a symplectic and $S := \text{Re}\langle\!\langle \cdot, \cdot \rangle\!\rangle$ a Riemannian structure on $\mathbb{H}$ and for all $\varphi, \psi \in \mathbb{H}$:

$$\langle\!\langle \varphi, \psi \rangle\!\rangle = S(\varphi, \psi) + i\Omega(\varphi, \psi) \qquad \text{and} \qquad \Omega(i\varphi, \psi) = S(\varphi, \psi)$$

i.e., $\mathbb{H}$ is a Kaehlerian manifold. Denote by $\cdot^{\perp}$ the orthogonal complement of $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ and for $\varphi \in \mathbb{H}$ define $\hat{\varphi} := \mathbb{C} \cdot \varphi$ and $\varphi^{\perp} := (\dot{\mathbb{C}} \cdot \varphi)^{\perp}$.

The *projective Hilbert space* $\hat{\mathbb{H}}$ of $\mathbb{H}$ is defined by $\hat{\mathbb{H}} := \dot{\mathbb{H}} / \dot{\mathbb{C}}$. $\hat{\mathbb{H}}$ is a complex analytic manifold, charts are given by the *projective representation:* For $\varphi_0 \in \dot{\mathbb{H}}$ define $\phi_{\varphi_0} : (\mathbb{C} \cdot \varphi_0)^{\perp} \to \hat{\mathbb{H}} \backslash \widehat{(\mathbb{C} \cdot \varphi_0)^{\perp}}, \varphi \mapsto \mathbb{C} \cdot (\varphi + \varphi_0)$. Then $\{(\hat{\mathbb{H}} \backslash \widehat{(\mathbb{C} \cdot \varphi_0)^{\perp}}, \phi_{\varphi_0}^{-1}) \mid \varphi_0 \in \dot{\mathbb{H}}\}$ is an *holomorphic atlas of* $\hat{\mathbb{H}}$. For $\hat{\varphi}_0 \in \hat{\mathbb{H}}$ *the tangent space* $T_{\hat{\varphi}_0} \hat{\mathbb{H}}$ is given by $T_{\hat{\varphi}_0} \hat{\mathbb{H}} = \{\varphi_0\} \times (\mathbb{C} \cdot \varphi_0)^{\perp} \subset \mathbb{H} \times \mathbb{H}$. Denote by $\dot{\pi} : \dot{\mathbb{H}} \to \hat{\mathbb{H}}$ the canonical projection. Then $(\dot{\mathbb{H}}, \dot{\pi}, \hat{\mathbb{H}})$ is a principal $\dot{\mathbb{C}}$ bundle. Note that $T\dot{\mathbb{H}} = \dot{\mathbb{H}} \times \mathbb{H}$.

A *connection form* α on $(\dot{\mathbb{H}}, \dot{\pi}, \hat{\mathbb{H}})$ is given by

$$\alpha(v_\varphi) := \frac{\langle\!\langle v, \varphi \rangle\!\rangle}{\langle\!\langle \varphi, \varphi \rangle\!\rangle} \qquad \text{for } v_\varphi \in T\dot{\mathbb{H}} = \dot{\mathbb{H}} \times \mathbb{H} \qquad (\text{i.e., } v \in \mathbb{H})$$

The corresponding decomposition of $T\dot{\mathbb{H}}$ is

$$T\dot{\mathbb{H}} \cong \dot{\mathbb{H}} \times \mathbb{H} \cong \text{Ver}\,\dot{\mathbb{H}} \times \text{Hor}\,\dot{\mathbb{H}} \cong \dot{\mathbb{H}} \times \mathbb{C} \times \text{Hor}\,\dot{\mathbb{H}}$$

$$v \mapsto (\varphi, v) \mapsto (\text{pr}_{\mathbb{C} \cdot \varphi}(v), \text{pr}_{(\mathbb{C} \cdot \varphi)^{\perp}}(v)) \mapsto \left(\varphi, \frac{\langle\!\langle v, \varphi \rangle\!\rangle}{\langle\!\langle \varphi, \varphi \rangle\!\rangle}, \text{pr}_{(\mathbb{C} \cdot \varphi)^{\perp}}(v)\right)$$

where pr_U is the orthogonal projection onto the subspace U. Define the following tensors on $\dot{\mathbb{H}}$:

$$\tilde{s}(v_\varphi, w_\varphi) := \frac{S(\text{pr}_{(\mathbb{C} \cdot \varphi)^{\perp}}(v), \text{pr}_{(\mathbb{C} \cdot \varphi)^{\perp}}(w))}{\langle\!\langle \varphi, \varphi \rangle\!\rangle} \qquad \text{for } v_\varphi, w_\varphi \in T\dot{\mathbb{H}}$$

$$\tilde{\omega}(v, w_\varphi) := \frac{1}{\langle\!\langle \varphi, \varphi \rangle\!\rangle} \cdot \Omega(\text{pr}_{(\mathbb{C} \cdot \varphi)^{\perp}}(v), \text{pr}_{(\mathbb{C} \cdot \varphi)^{\perp}}(w))$$

$$\tilde{j}(v_\varphi) := \frac{1}{\langle\!\langle \varphi, \varphi \rangle\!\rangle} J(v_\varphi) = \frac{1}{\langle\!\langle \varphi, \varphi \rangle\!\rangle} \cdot iv$$

Then $\tilde{s}, \tilde{\omega}, \tilde{j}$ are $\dot{\mathbb{C}}$ invariant. $\tilde{s}$ and $\tilde{\omega}$ are horizontal and $\tilde{j}(\text{Hor}\,\dot{\mathbb{H}}) \subset \text{Hor}\,\dot{\mathbb{H}}$. So $\tilde{s}, \tilde{\omega}, \tilde{j}$ induce $\dot{\pi}$-related tensors s, ω, j on $\hat{\mathbb{H}}$ with $\omega(j \cdot, \cdot) = s(\cdot, \cdot)$.

5.1. *Lemma.* Define the 1-form β on $\dot{\mathbb{H}}$ by $\beta(v_\varphi) = \Omega(v, \varphi)/\langle\!\langle \varphi, \varphi \rangle\!\rangle$. Then $d\beta = -\tilde{\omega}$.

Proof.

$$d\beta_\varphi = d\left(\frac{1}{\langle\!\langle \varphi, \varphi \rangle\!\rangle}\right) \wedge \Omega(\cdot, \varphi) - \frac{1}{\langle\!\langle \varphi, \varphi \rangle\!\rangle} \Omega$$

$$= -\frac{1}{\langle\!\langle \varphi, \varphi \rangle\!\rangle} \left(\Omega + 2\,\frac{s(\cdot, \varphi) \wedge \Omega(\cdot, \varphi)}{\langle\!\langle \varphi, \varphi \rangle\!\rangle}\right) = -\tilde{\omega}_\varphi$$

since

$$\mathrm{Re}\langle\!\langle \cdot\,, \varphi\rangle\!\rangle \wedge \mathrm{Im}\langle\!\langle \varphi, \cdot\,\rangle\!\rangle = \tfrac{1}{2}\langle\!\langle \varphi, \varphi\rangle\!\rangle \cdot \Omega|_{\varphi^\perp \times \varphi^\perp}$$

So $\tilde{\omega}$ is exact and therefore ω closed and we obtain the following:

5.2. *Proposition.* Let $\mathbb{H}$ be a complex Hilbert space. Then $\hat{\mathbb{H}} = \dot{\mathbb{H}}/\dot{\mathbb{C}}$ becomes a Kaehlerian manifold in a natural way. In particular $(\hat{\mathbb{H}}, \omega)$ is a symplectic manifold.

Denote by $\mathbb{H}^c := \dot{\mathbb{H}}/\mathbb{R}^+$ and $p\colon \dot{\mathbb{H}} \to \mathbb{H}^c$ the natural projection. Then $\mathbb{H}^c$ is a principal circle bundle over $\hat{\mathbb{H}}$. Define $\pi^c\colon \mathbb{H}^c \to \mathbb{H}$ by $\pi^c = \dot{\pi}|_{\mathbb{H}^c}$. The bundle $(\dot{\mathbb{H}}, p, H^c)$ is a principal $(\mathbb{R}^+, \cdot\,)$ bundle and we can identify $\mathbb{H}^c$ with the one-sphere of $\mathbb{H}$ $S^1(\mathbb{H}) = \{\varphi \in \mathbb{H} \mid \langle\!\langle \varphi, \varphi\rangle\!\rangle = 1\}$.

The one-form β on $\dot{\mathbb{H}}$ defined by 5.1 is $\mathbb{R}^+$ invariant and horizontal, so there exists an unique one-form α^c on $\mathbb{H}^c$ with $p^*\alpha^c = \beta$. Moreover, we get by differentiation the following:

5.3. *Lemma.* α^c is a connection form of the bundle $(\mathbb{H}^c, \pi^c, \hat{\mathbb{H}}, \mathbb{T})$ and $\alpha = d(\log \langle\!\langle \cdot\,, \cdot\,\rangle\!\rangle^{1/2}) + i \cdot p^*\alpha^c$

5.4. *Theorem.* Let $(\mathbb{H}, \langle\!\langle \cdot\,, \cdot\,\rangle\!\rangle)$ be a complex Hilbert space. Then $(\mathbb{H}^c, \alpha^c)$ is a prequantum bundle over the projective Hilbert space $\hat{\mathbb{H}} = \dot{\mathbb{H}}/\dot{\mathbb{C}}$. The bundle $(\dot{\mathbb{H}}, \dot{\pi}, \hat{\mathbb{H}})$ is the corresponding complex PQB.

Proof. We have only to show that $\mathrm{curv}\,\alpha^c = -\omega$. By 5.3 $\mathrm{curv}\,\alpha^c = \mathrm{curv}\,\alpha$ and by 5.1 $dp^*\alpha^c = d\beta = -\tilde{\omega}$. So $\dot{\pi}^*\omega = -p^*d\alpha^c = -d\alpha$.

Note. Any Γ sphere for $\Gamma \in \mathbb{R}^+$ of $\mathbb{H}$ is a PQB over $\hat{\mathbb{H}}$.

Caution! The corresponding line bundle $\dot{\mathbb{H}} \times_{\dot{\mathbb{C}}} \mathbb{C}$ is *not* the Hilbert space $\mathbb{H}$. $\mathbb{H}$ is not even a bundle over $\hat{\mathbb{H}}$.

5.5. *Proposition.* The induced symplectic structure (see 4.9.) on the total space of the complex prequantum bundle $(\dot{\mathbb{H}}, \pi, \hat{\mathbb{H}})$ coincides with the natural symplectic structure Ω on $\mathbb{H}$.

Proof.

$$d(\langle\!\langle \varphi, \varphi\rangle\!\rangle \cdot p^*\alpha^c) = (d\langle\!\langle \varphi, \varphi\rangle\!\rangle) \wedge p^*\alpha^c + \langle\!\langle \varphi, \varphi\rangle\!\rangle \cdot p^*d\alpha^c$$

$$= 2S(\varphi, \cdot\,) \wedge \frac{\Omega(\cdot\,, \varphi)}{\langle\!\langle \varphi, \varphi\rangle\!\rangle} - \langle\!\langle \varphi, \varphi\rangle\!\rangle \frac{\Omega|_{(\mathbb{C}\cdot\varphi)^\perp \times (\mathbb{C}\cdot\varphi)^\perp}}{\langle\!\langle \varphi, \varphi\rangle\!\rangle} = -\Omega$$

(Compare the proof of 5.1.)

5.6. *Corollary.* The Kaehler structure on $\dot{\mathbb{H}}$ is induced by the Kaehler structure on $\hat{\mathbb{H}}$, the base of the complex PQB $(\dot{\mathbb{H}}, \dot{\pi}, \hat{\mathbb{H}})$.

5.7. *Proposition.* Let $X \in \mathscr{X}\dot{\mathbb{H}}$ be a vector field, i.e., let X be a smooth (not necessary linear) operator $\dot{\mathbb{H}} \to \mathbb{H}$. Then $X \in \mathscr{H}^{\mathrm{inv}}\,\dot{\mathbb{H}} = \mathscr{P}\dot{\mathbb{H}}$ iff

(a) $X(c \cdot \varphi) = c \cdot X(\varphi)$ for all $\varphi \in \dot{\mathbb{H}}$ and $c \in \dot{\mathbb{C}}$ and iff

(b) $\langle\!\langle X(\varphi), \psi\rangle\!\rangle + \langle\!\langle \varphi, X(\psi)\rangle\!\rangle = 0$ for all $\varphi, \psi \in \dot{\mathbb{H}}$. (i.e., X is anti-symmetric.)

Proof. X is $\dot{\mathbb{C}}$-invariant iff $c^*X = X$. $c^*X(\varphi) = Tc^{-1} \circ X \circ c(\varphi) = c^{-1} \cdot X(c \cdot \varphi)$. For all $c \in \dot{\mathbb{C}}$ and $\varphi \in \dot{\mathbb{H}}$. This proves (a). Let $F_t: I_\epsilon \times U \to \dot{\mathbb{H}}$ be the (local) flow of X. Then F_t is $\dot{\mathbb{C}}$-equivariant iff X is $\dot{\mathbb{C}}$ invariant. F_t is isometric iff for all $\varphi, \psi \in \dot{\mathbb{H}}$ $0 = (d/dt) \langle\langle F_t\varphi, F_t\psi \rangle\rangle = \langle\langle F_t\varphi, \psi \rangle\rangle + \langle\langle \varphi, F_t\psi \rangle\rangle$, and any $\dot{\mathbb{C}}$-equivariant flow is isometric iff it is symplectic.

Remark. Because of 5.7 (a) any invariant vector field $X \in \mathscr{X}^{\mathrm{inv}} \dot{\mathbb{H}}$ extends to a smooth operator $X: \mathbb{H} \to \mathbb{H}$ by $X(0) = 0$.

5.8. *Remark.* $\pi_1(\hat{\mathbb{H}}) = 0$, so $\mathrm{Spl}(\hat{\mathbb{H}}) = \mathrm{Spl}^{\mathbb{H}^c}(\hat{\mathbb{H}})$ and $(\mathbb{H}^c, \alpha^c)$ is the unique PQB over $(\hat{\mathbb{H}}, \omega)$ (for dim $\mathbb{H} \neq 1$).

6. *On Quantum Dynamics*

In this section we want to give a short outline of how prequantum bundles can be used to describe the dynamics of a general quantum system. For the foundations of quantum mechanics we refer to Jauch (1968). Our basic assumptions are as follows:

> A quantum mechanical system is described by a complex Hilbert space $\mathbb{H}$. The lattice $\mathscr{L}(\mathbb{H})$ of all closed subspaces of $\mathbb{H}$ is the *logic* of the quantum system. Elements of the projective space $\hat{\mathbb{H}}$ represent the *pure states. Observables* are represented by the (possibly unbounded) self-adjoint operators of $\mathbb{H}$.

The unitary operators are the isomorphisms of the system. For simplicity we will restrict our attention in the following to smooth operators. All results of this section can be extended to unbounded operators by using the methods of Marsden (1968).

Notation.

$U(\mathbb{H})$:= the Banach Lie group of unitary operators
$\mathrm{Proj}(\mathbb{H})$:= $U(\mathbb{H})/\mathbb{T}$, the projective group
$\mathscr{S}A(\mathbb{H})$:= the symmetric operators of $\mathbb{H}$
$i\mathscr{S}A(\mathbb{H})$:= the Lie algebra of antisymmetric operators
$\mathscr{P}j(\mathbb{H}) := i\,\mathscr{S}A(\mathbb{H})/\mathbb{R}$ ($X \sim Y$ iff $X - Y = i \cdot \lambda \cdot 1$ for $X, Y \in i\,\mathscr{S}A(\mathbb{H})$)

$i\,\mathscr{S}A(\mathbb{H})$ and $\mathscr{P}j(\mathbb{H})$ are the Lie algebras of $U(\mathbb{H})$ and $\mathrm{Proj}(\mathbb{H})$. By definition the following sequences of groups or, respectively, Lie algebras are exact:

$$0 \to \mathbb{T} \hookrightarrow U(\mathbb{H}) \to \mathrm{Proj}(\mathbb{H}) \to 0$$
$$0 \to \mathbb{R} \hookrightarrow i\,\mathscr{S}A(\mathbb{H}) \to \mathscr{P}j(\mathbb{H}) \to 0$$

Note. Elements of $\mathrm{Proj}(\mathbb{H})$ project to morphisms of $\hat{\mathbb{H}}$ and elements of $\mathscr{P}j(\mathbb{H})$ to vector fields of $\hat{\mathbb{H}}$.

The following proposition is well known (Marsden, 1968):

6.1. *Proposition.* Let X be a complex linear vector field on $\mathbb{H}$ with complex linear flow F_t. (I.e., X is a linear operator.) Then the following are equivalent:

(i) X is Hamiltonian
(ii) $X \lrcorner \Omega = dE$ with $E\colon \mathbb{H} \to \mathbb{R}, E(\varphi) = \frac{1}{2}\langle\langle iX\varphi, \varphi\rangle\rangle$
(iii) $X \in i\,\mathcal{S}A(\mathbb{H})$, i.e., iX is symmetric
(iv) F_t is symplectic for all $t \in \mathbb{R}$
(v) F_t is unitary for all $t \in \mathbb{R}$.

The *dynamics* of the quantum system is given by the *Schrödinger operator* $\mathrm{H} \in \mathcal{S}A(\mathbb{H})$. For $\hbar \in \mathbb{R}^+$ we have $\mathrm{H} = i\hbar \cdot X$ for a complex linear Hamiltonian vector field X on $\mathbb{H}$. Integral curves of X are solutions of the *Schrödinger equation* and describe the time evolution of the system:

$$i\hbar \left.\frac{d\varphi_t}{dt}\right|_{t=0} = \mathrm{H}(\varphi)$$

Now consider the PQB $(\dot{\mathbb{H}}, \alpha)$ over $(\hat{\mathbb{H}}, \omega)$. Then we obtain by 5.7. the following:

6.2. *Lemma:*

$$U(H) = \{F \in \mathrm{Preq}(\dot{\mathbb{H}}) \mid F \text{ is additive}\} \subset \mathrm{Preq}(\dot{\mathbb{H}}, \alpha)$$

$$i\,\mathcal{S}A(\mathbb{H}) = \{X \in \mathcal{P}(H) \mid X \text{ is additive}\} \subset \mathcal{P}\dot{\mathbb{H}}$$

So $U(\dot{\mathbb{H}})$ consists of all prequantum-morphisms, which map closed subspaces of $\mathbb{H}$ into closed subspaces, i.e., which are also morphisms of the logic $\mathcal{L}(\mathbb{H})$.

6.3. *Corollary.* The following diagrams of groups or, respectively, Lie algebras are commutative with exact lines:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{T} & \hookrightarrow & \mathrm{Preq}(\dot{\mathbb{H}}, \alpha) & \longrightarrow & \mathrm{Spl}(\hat{\mathbb{H}}) & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathbb{T} & \hookrightarrow & U(\mathbb{H}) & \longrightarrow & \mathrm{Proj}(\mathbb{H}) & \longrightarrow & 0 \end{array} \qquad (6.1)$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R} & \hookrightarrow & \mathcal{P}\dot{\mathbb{H}} & \longrightarrow & \mathcal{H}\hat{\mathbb{H}} & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathbb{R} & \hookrightarrow & i\,\mathcal{S}A(\mathbb{H}) & \longrightarrow & \mathcal{P}j\mathcal{H} & \longrightarrow & 0 \end{array} \qquad (6.1')$$

Using the results of Sections 4 and 5 we obtain the following (formal) correspondence between the structure of a (time-independent) classical mechanical system and a quantum mechanical system:

Classical system	*Quantum system*
Phase space (M, ω)	Space of pure states $(\hat{\mathbb{H}}, \omega)$
Hamiltonian systems $\mathscr{H}M$	Hamiltonian Systems on $\hat{\mathbb{H}}: \mathscr{H}\mathbb{H}$
Logic: $\mathscr{B}(M)$ (Borel sets of M)	Logic: $\mathscr{L}(\mathbb{H}) \subset \mathscr{B}(\hat{\mathbb{H}})$
Observables: $\mathscr{F}M$	$i\mathscr{S}A(\mathbb{H}) \subset \mathscr{P}\dot{\mathbb{H}} \cong \mathscr{F}\hat{\mathbb{H}}$
Canonical transformations: $\mathrm{Spl}(M, \omega) = \mathrm{Spl}(M, \omega) \cap \mathrm{Aut}\mathscr{B}(M)$	$\mathrm{Proj}(\mathbb{H}) \cong \mathrm{Spl}(\hat{\mathbb{H}}, \omega) \cap \mathrm{Aut}\,\mathscr{L}(\mathbb{H})$
Hamiltonian function $f \in \mathscr{F}M$	Expectation value $\hat{E}: \dot{\mathbb{H}} \to \mathbb{R}$ $$\hat{E}(\varphi) = \frac{\langle\langle \mathsf{H}\varphi, \varphi\rangle\rangle}{2\hbar\langle\langle \varphi, \varphi\rangle\rangle}$$
Hamiltonian vector field X_f	$X_{\hat{E}} = d\hat{E}^{\#}$ on $\hat{\mathbb{H}}$ (H Schrödinger Op.)
PQB $(\dot{L}, \alpha)$ over M (if ω is integral)	Hilbert–PQB $(\dot{\mathbb{H}}, \alpha)$
Spl form of $\dot{L}: \Omega$ (see 4.9)	$\Omega = \mathrm{Im}\langle\langle \cdot\,, \cdot\rangle\rangle$ (see 5.5)
P_f	$P_{\hat{E}} = X_E = \dfrac{1}{i\hbar}\,\mathsf{H}$
Hamiltonian of P_f: $\tilde{f} = s^2 \cdot f \circ \dot{\pi}$ (see 4.12)	Hamiltonian of X_E: $E = \langle\langle \cdot\,, \cdot\rangle\rangle \cdot \hat{E},\ \varphi \mapsto (1/2\hbar) \cdot \langle\langle \mathsf{H}\varphi, \varphi\rangle\rangle$
Time evolution: $F_\cdot$ flow of X_f: $$\frac{d}{dt} F_t(m)\vert_{t=0} = X_f(m),$$ $m \in M$	$\hat{\phi}$ flow of $X_{\hat{E}}$ iff $$\frac{d}{dt}\hat{\phi}_t(\hat{\varphi})\vert_{t=0} = X_{\hat{E}}(\hat{\varphi})$$
$\tilde{F}$ is $\dot{\pi}$ related to F with $$\frac{d}{dt}\tilde{F}_t(x_m)\vert_{t=0} = P_f(x_m)$$ (see 4.12)	ϕ is $\dot{\pi}$-related to $\hat{\phi}$ with $$i\hbar \cdot \frac{d}{dt}\phi_t(\varphi)\vert_{t=0} = i\hbar \cdot X_E(\varphi) = \mathsf{H}(\varphi)$$ (Schrödinger equation)

7. *The Prequantization Functor*

The main application of prequantum bundles is the prequantization procedure of B. Kostant (1970). Using the Hilbert PQB of Section 5 we give a functorial definition of prequantization.

Denote by $\mathbb{T}$-PQB the category of prequantum bundles with morphisms as defined in 4.3. By the results of Section 4 $\mathbb{T}$-PQB is equivalent to the category $\dot{\mathbb{C}}$-PQB of all complex prequantum bundles and to the category L-PQB of all line bundles with covariant derivative ∇, ∇-affine Hermitian structure and symplectic curvature. The subcategories of finite dimensional prequantum bundles will be denoted, respectively, by $\mathbb{T}$-PQB^f, $\dot{\mathbb{C}}$-PQB^f, L-PQB^f.

Let $L_i^c = (L_i^c, \alpha_i^c, M_i) \in \mathbb{T}$-PQB^f and $L_i = (L_i, \nabla^i, \langle\cdot\,,\cdot\rangle_i)$ the corresponding line bundles of L-PQB^f, $i = 1, 2$. For any PQB morphism $f^c: L_1^c \to L_2^c$ the induced

$$\begin{array}{ccc} L_1 & \xrightarrow{f} & L_2 \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow{\hat{f}} & M_2 \end{array}$$

L-PQB morphism $f: L_1 \to L_2$ restrict on each fiber of L_1 to an isomorphism $f_m: \pi_1^{-1}(m) \to \pi_2^{-2}(\hat{f}(m))$. f is a PQB isomorphism iff the base map $\hat{f}: M_1 \to M_2$ is a symplectomorphism.

Therefore we can define the contravariant section functor Γ on L-PQB: $\Gamma(L) := \{\sigma: M \to L \mid \sigma \text{ section}\}$ and $\Gamma(f) := f^*: \Gamma(L_2) \to \Gamma(L_1), f^*\sigma(m) := f_m^{-1}(\sigma[f(m)])$ for $f: L_1 \to L_2, m \in M_1, \sigma \in \Gamma(L_2)$.

For any finite-dimensional PQB (L^c, α^c) over (M, ω) the form $\rho = \omega^{(\dim M)/2}$ is a natural volume on M (the *Liouville measure*). Using this volume and the affine Hermitian structure we can define on the category L-PQB^f the Hilbert space valued L^2-section functor

$$\Gamma^{L^2}(L) := \{\sigma \in \Gamma(L) \mid \textstyle\int_M \langle \sigma(m), \sigma(m) \rangle \rho(m) \in \mathbb{R}\}/N =: \mathbb{H}(L)$$

$$\Gamma^{L^2}(f) := f^* \text{ [restricted on } \mathbb{H}(L_2)]$$

for $L = (L, \nabla, \langle \cdot , \cdot \rangle) \in L\text{-PQB}^f$ over (M, ω) and $f: L_1 \to L_2$. (N denotes the space of all sections, which vanish except on a set of measure zero.)

7.1. *Proposition:* Let $(L_i, \nabla^i, \langle \cdot , \cdot \rangle_i) \in L\text{-PQB}^f$, $i = 1, 2$ and $f: L_1 \to L_2$ a PQB morphism. Then $f^*: \mathbb{H}(L_2) \to H(L_1)$ is a partial isometry, i.e., $f^*|_{(\mathrm{Ker} f^*)^\perp}: (\mathrm{Ker} f^*)^\perp \to \mathbb{H}(L_1)$ is an isometry.

Proof. For the Hermitian structure in $\mathbb{H}(L_1)$ we have

$$\langle\!\langle f^*\sigma, f^*\sigma \rangle\!\rangle_1 = \textstyle\int_{M_1} \langle f_m^{-1} \circ \sigma \circ \hat{f}(m), f_m^{-1} \circ \sigma \circ \hat{f}(m) \rangle \rho_1(m)$$

$$= \textstyle\int_{\hat{f}(M_1)} \langle \sigma(m), \sigma(m) \rangle \cdot \rho_2|_{\hat{f}(M_2)}(m)$$

since f is symplectic and therefore volume preserving.

$$\mathrm{Ker} f^* = \{\sigma \in \mathbb{H}(L_2) \mid \sigma \circ \hat{f} = 0\} = \{\sigma \in \mathbb{H}(L_2) \mid \sigma|_{\hat{f}(M_1)} = 0\}$$

$$(\mathrm{Ker} f^*)^\perp = \{\sigma \in \mathbb{H}(L_2) \mid \sigma|_{M_2 \setminus \hat{f}(M_1)} = 0\}$$

So $\langle\!\langle \sigma, \sigma \rangle\!\rangle_2 = \langle\!\langle f^*\sigma, f^*\sigma \rangle\!\rangle_1$ iff $\sigma \in (\mathrm{Ker} f^*)^\perp$.

7.2 *Definition.* $K(f): \mathbb{H}(L_1) \to \mathbb{H}(L_2), K(f) := (f^*|_{(\mathrm{Ker} f^*)^\perp})^{-1}$

$K(f)$ is a $\dot{\mathbb{C}}$-linear isometric imbedding $\mathbb{H}(L_1) \hookrightarrow \mathbb{H}(L_2)$ and therefore a symplectic map. Let $\dot{K}(f)$ be $K(f)$ restricted on $\dot{\mathbb{H}}(L_1) = \mathbb{H}(L_1) \setminus \{0\}$. Then $\dot{K}(f)$ project to a symplectic map $\widehat{K(f)}: \hat{\mathbb{H}}(L_1) \to \hat{\mathbb{H}}(L_2)$, so $\dot{K}(f): (\dot{\mathbb{H}}_1, \dot{\pi}_{\mathbb{H}_1}, \hat{\mathbb{H}}_1) \to (\dot{\mathbb{H}}_2, \dot{\pi}_{\mathbb{H}_2}, \hat{\mathbb{H}}_2)$ with $\mathbb{H}_i := \mathbb{H}(L_i)$ is a PQB morphism, Moreover, $K(f \circ g) = K(f) \circ K(g)$ for PQB morphisms f and g. This gives the following:

7.3. *Proposition.* $\dot{K}: L\text{-PQB}^f \to \dot{\mathbb{C}}\text{-PQB}$ is a covariant functor. By equivalence of L-PQB and $\dot{\mathbb{C}}$-PQB $\dot{K}$ is also a covariant functor $\dot{\mathbb{C}}\text{-PQB}^f \to \dot{\mathbb{C}}\text{-PQB}$.

7.4. *Definition.* The functor $K: \dot{\mathbb{C}}\text{-PQB}^f \to \dot{\mathbb{C}}\text{-PQB}$ will be called the *Kostant functor.*

Remark. If $f_i: L \to L', i = 1, 2$ and $L, L' \in L\text{-PQB}^f$ are PQB morphisms over the same base map $g: M \to M'$ then by 4.4 $f_1 = c \cdot f_2$ for a $c \in \mathbb{T}$, and so $\widehat{K(f_1)} = \widehat{K(f_2)}$, i.e., the induced symplectic map $\hat{\mathbb{H}}(L) \to \hat{\mathbb{H}}(L')$ only depends on the base map g and—of course—on L and L'. Define $\hat{K}(g) := \widehat{K(f_1)} = \widehat{K(f_2)}$. This definition depends on the bundles L and L', therefore $\hat{K}$ is *not* a functor. We obtain by 4.2 and 4.3 only for the simply connected case the following:

> 7.5. *Proposition.* The Kostant functor K induces a covariant functor $\hat{K}$ from the category of simply connected finite-dimensional symplectic manifolds (M, ω) with integral ω into the category of symplectic manifolds by $\hat{K}(M, \omega) := \widetilde{K(L)}$ and $\hat{K}(g) := \hat{K}(f)$, where L is the unique PQB over (M, ω) and f is a PQB morphism over g.

Let $(\dot{L}, \alpha)$ be a PQB^f over (M, ω). Then $K\colon \operatorname{Preq}(\dot{L}, \alpha) \to \operatorname{Preq}(\dot{\mathbb{H}}(L), \alpha_{\mathbb{H}})$ is a group morphism. The induced infinitesimal morphism will be the prequantization map of Kostant. To prove this we need the following:

> 7.6. *Lemma.* Let $(\dot{L}, \alpha)$ be a PQB over (M, ω) and $L = (L, \nabla, \langle\cdot,\cdot\rangle)$ the corresponding line bundle, $q\colon \dot{L} \times \mathbb{C} \to L$
>
> $$\begin{array}{ccc} \dot{L}\times\mathbb{C} & \xrightarrow{q} & L \\ \downarrow & & \downarrow \\ \dot{L} & \longrightarrow & M \end{array}$$
>
> the natural projection and $q^*\colon \Gamma^\infty L \to \mathscr{F}^{\mathrm{equ}}(\dot{L}, \mathbb{C})$ the induced isomorphism. Then for any $f \in \mathscr{F}M$ and $\sigma \in \Gamma^\infty L$
>
> $$\mathbb{L}_{P_f} q^*\sigma = q^*(\nabla_{df^\#}\sigma + i\hbar f\cdot\sigma)$$

Proof. $P_f = G_f + f \circ \dot{\pi} \cdot Z = G_f + i\hat{\hbar} \cdot f \circ \dot{\pi}$ by 3.10.

$$\mathbb{L}_{i\hat{\hbar}\cdot f\circ\pi} q^*\sigma = i\hbar \cdot f \circ \pi \cdot \mathbb{L}_{\hat{1}} q^*\sigma = i\hbar \cdot f \circ \dot{\pi} \cdot q^*\sigma = q^*(i\hbar \cdot f \cdot \sigma)$$

since $q^*\sigma$ is $\dot{\mathbb{C}}$ equivariant. $\mathbb{L}_{G_f} q^*\sigma = q^*(\nabla_{df}\#\sigma)$ by definition of ∇ (see Greub et al., 1973). Let $\Gamma_c^\infty(L)$ be the space of all C^∞ sections with compact support. $\Gamma_c^\infty(L)$ is a dense subset of $\mathbb{H}(L) =: \mathbb{H}$, so the map $\mathscr{F}M \cong \mathscr{P}L \to \operatorname{End}\Gamma^\infty L$, $f \mapsto \nabla_{df}\# + i\hbar \cdot f$ defines a map $k\colon \mathscr{F}M \to \operatorname{End}\mathbb{H}$. If $i\tilde{\mathscr{S}}A(\mathbb{H})$ is the space of all—possibly unbounded—antisymmetric $\mathbb{C}$-linear operators of $\mathbb{H}$, then $k(f) \in i\tilde{\mathscr{S}}A(\mathbb{H})$ (since the Hermitian structure $\langle\cdot,\cdot\rangle$ on L is ∇ affine). Moreover, for all $f, g \in \mathscr{F}M$: $k(\{f, g\}) = [k(f), k(g)]$ (commutator) by 7.6 and 2.6. So we have the following:

> 7.7. *Proposition:* $k\colon \mathscr{F}M \to i\tilde{\mathscr{S}}A(\mathbb{H}), f \mapsto \nabla_{df}\# + i\hbar \cdot f$ is a Lie algebra homomorphism induced by the group morphism $K\colon \operatorname{Preq}(\dot{L}, \alpha) \to \operatorname{Preq}(\dot{\mathbb{H}})$. In particular $k(f) \in \mathscr{P}\dot{\mathbb{H}}$ if $k(f)$ is bounded. k is called the *Kostant prequantization map.*

Acknowledgments

I wish to thank the members of the group "Geometric Quantization" at the Technische Universität Berlin under guidance of K. E. Hellwig for helpful discussions. I should also like to thank K. E. Hellwig, H. Heß, and R. Zavodnik for reading the manuscript and for critical remarks. This work is a part of the author's thesis at the Freie Universität Berlin.

References

Abraham, R. and Marsden, J. E. (1967). *Foundations of Mechanics*, Benjamin, Reading, Massachusetts (1972).

Flaschel, P. and Klingenberg, W. (1972). *Riemannsche Hilbertmannigfaltigkeiten, Periodische Geodätische*, Springer, Lecture Notes in Mathematics 282.

Greub, W., Halperin, S. and Vanstone, R. (1973). *Connections, Curvature and Cohomology II*, Academic Press, New York/London.

Gawedzki, K. and Szapiro, T. (1974). *Reports on Mathematical Physics*, **6**, 477.

Hermann, R. (1973). *Topics in the Mathematics of Quantum Mechanics* (Interdisciplinary Mathematics VI), privately printed, U.S.A.

Jauch, J. M. (1968). *Foundations of Quantum Mechanics*, Addison-Wesley, Reading, Massachusetts.

Kostant, B. (1970). "Quantization and Unitary Representations," in: *Lectures in Modern Analysis and Applications III*, Taam, C. T. ed., Springer Lecture Notes in Mathematics, 170.

Kostant, B. (1973). Line Bundles and the Prequantized Schrodinger Equation, *Symposia Math.*

Marsden, J. E. (1968). *Archive for Rational Mechanics and Analysis*, **28**, 362.

Souriaw, J. (1970). *Structure des Systemes Dynamiques*, Dunod, Paris.

Vaisman, I. (1973). *Cohomology and Differential Forms*, Marcel Dekker, Inc., New York.